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PLASTICITY THEORY FORMULATED IN TERMS OF PHYSICALLY BASED MICROSTRUCTURAL VARIABLES—PART II. EXAMPLES

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Abstract—Specific constitutive equations for an elastically isotropic material with orthotropic plastic relaxation are developed within the context of the general theory proposed in the companion paper. Analytical expressions for the steady-state solution to uniaxial stress are developed to show the effect of orthotropic plastic relaxation. Numerical simulations of large deformation simple shear and isochoric extension are used to explore various aspects of the material response. In particular, the texturing effect caused by the rotation of the triad **m**, which is used to characterize the average atomic lattice is examined.

1. INTRODUCTION

In a companion paper (Rubin, 1994) general constitutive equations for anisotropic elastic–plastic materials are developed using physically based microstructural variables. Specifically, a vector triad $\mathbf{m}_i$ is introduced which characterizes the orientation and elastic deformation of the average atomic lattice relative to a Reference Lattice State (RLS) associated with the material when it is stress-free and at specified reference temperature θ_0. For convenience, the reader is referred to the companion paper (Rubin, 1994) for definitions and equations that are not repeated here.

The main objective of this part of the present work is to develop specific constitutive equations for an elastic–viscoplastic material that exhibits isotropic elastic response but may exhibit directional plastic response. Of particular interest is the constitutive equation for plastic spin $\mathbf{W}_p$ which will be used to orient the vector triad $\mathbf{m}_i$ relative to the deformation process. To this end, the work of Raniecki and Mroz (1990) is followed and it is assumed that whenever the material is subjected to a constant symmetric velocity gradient

$$\mathbf{L} = \mathbf{D} = \text{constant}, \tag{1}$$

the lattice tends to rotate and orient itself with the principal directions of **D**. Within the context of the present theory this means that asymptotically the vectors $\mathbf{m}_i$ align themselves with these principal directions. This idea will be used to specify a simple constitutive equation for the plastic spin $\mathbf{W}_p$. However, one should not confuse $\mathbf{m}_i$ with normals to slip systems of plastic deformation which would tend to align themselves with the planes of maximum shear. Furthermore, it is emphasized that when **L** is constant but not symmetric, the vectors $\mathbf{m}_i$ will not necessarily be orthogonal vectors and thus, they cannot align themselves exactly with the principal directions of **D**.

In the following sections the specific constitutive equations are developed and analytical expressions for the steady-state solution to uniaxial stress, which shows the effect of orthotropic plastic relaxation, are derived. Next, numerical examples of simple shear and isochoric extension are considered to explore various aspects of the material response. In particular, the texturing effect caused by the rotation of the triad $\mathbf{m}_i$ is examined.

2. SPECIFIC CONSTITUTIVE EQUATIONS

General thermomechanical forms for the Helmholtz free energy ψ and the entropy flux $\mathbf{p}$ have been considered for elastically isotropic elastic–viscoplastic materials in the works of Rubin (1987a,b; 1989). For the present purposes it is assumed that in the reference configuration the material is stress-free and at reference temperature θ_0 so that the reference value of the dilatation J is unity. Since the reference configuration is associated with a RLS it follows that

$$J^2 = J_{\mathrm{m}}^2 = m = \det m_{ij}, \quad m_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j, \tag{2a,b}$$

where m_{ij} is the metric associated with the variables $\mathbf{m}_i$. Furthermore, it is convenient to define distortional measures $\mathbf{m}_i'$ of the lattice by the formulas

$$\mathbf{m}_i' = m^{-1/6}\mathbf{m}_i, \quad m_{ij}' = \mathbf{m}_i' \cdot \mathbf{m}_j' = m^{-1/3} m_{ij}, \quad \det m_{ij}' = 1. \tag{3a,b,c}$$

Now, simple constitutive equations for ψ and $\mathbf{p}$ are proposed of the forms

$$2\rho_0\psi = 2\rho_0 C_{\mathrm{v}}\left[\theta - \theta_0 - \theta\ln\left(\frac{\theta}{\theta_0}\right)\right] + 6k\alpha(\theta-\theta_0)(1-J_{\mathrm{m}}) + 2k(J_{\mathrm{m}} - 1 - \ln J_{\mathrm{m}}) + \mu(\alpha_1 - 3), \tag{4a}$$

$$\mathbf{p} = -\frac{K_\theta}{\theta}\mathbf{g}, \tag{4b}$$

where ρ_0 is the mass density in the reference configuration, C_{v} is the specific heat at constant volume, α is the coefficient of linear expansion, k is the bulk modulus, μ is the shear modulus, α_1 is a pure measure of elastic distortion defined in Rubin (1994), $\mathbf{g} = \partial\theta/\partial\mathbf{x}$ is the temperature gradient relative to the present position $\mathbf{x}$ of a material point, and K_θ is the heat conduction coefficient. Also, the quantities C_{v}, α, k, μ, K_θ are assumed to be positive constants. It follows from Rubin (1994) and the assumptions (4) that the specific entropy η, the specific internal energy ε, the pressure p, the deviatoric Cauchy stress $\mathbf{T}'$, and the part ξ' of the rate of internal entropy production are given by

$$\rho_0\eta = \rho_0 C_{\mathrm{v}}\ln\left(\frac{\theta}{\theta_0}\right) - 3k\alpha(1-J_{\mathrm{m}}), \tag{5a}$$

$$2\rho_0\varepsilon = 2C_{\mathrm{v}}(\theta-\theta_0) - 6k\alpha\theta_0(1-J_{\mathrm{m}}) + 2k(J_{\mathrm{m}} - 1 - \ln J_{\mathrm{m}}) + \mu(\alpha_1 - 3), \tag{5b}$$

$$p = k\left(\frac{1}{J_{\mathrm{m}}} - 1\right) + 3k\alpha(\theta - \theta_0), \tag{5c}$$

$$\mathbf{T}' = \mu J_{\mathrm{m}}^{-1}(\mathbf{m}_r' \otimes \mathbf{m}_r' - \tfrac{1}{3}m_{rr}'\mathbf{I}), \tag{5d}$$

$$\rho\theta\xi' = \mathbf{T} \cdot \mathbf{D}_{\mathrm{p}}, \tag{5e}$$

where the condition (2a) and the conservation of mass in the form

$$\rho J = \rho_0, \tag{6}$$

have been used. Furthermore, it is noted (Rubin, 1994) that the second law of thermodynamics will be satisfied whenever

$$K_\theta > 0, \quad C_v > 0, \quad \mathbf{T}\cdot\mathbf{D}_p \geqslant 0. \tag{7a,b,c}$$

Here, attention is focused on the purely mechanical theory which can be obtained by taking $\theta = \theta_0$ so that the pressure p reduces to

$$p = k\left(\frac{1}{J_m} - 1\right). \tag{8}$$

Recall now from Rubin (1994) that the vectors $\mathbf{m}_i$ are determined by integrating evolution equations of the form

$$\dot{\mathbf{m}}_i = \mathbf{L}_m\mathbf{m}_i, \quad \mathbf{L}_m = \mathbf{L} - \mathbf{L}_p, \quad \mathbf{L}_p = \mathbf{D}_p + \mathbf{W}_p, \tag{9a,b,c}$$

where $\mathbf{L}_p$ requires a constitutive equation. It follows from (2), (3) and (9) that

$$\frac{\dot{J}_m}{J_m} = \mathbf{D}\cdot\mathbf{I}, \quad \mathbf{D}' = \mathbf{D} - \tfrac{1}{3}(\mathbf{D}\cdot\mathbf{I})\mathbf{I}, \tag{10a,b}$$

$$\dot{m}'_{ij} = 2(\mathbf{D}' - \mathbf{D}_p)\cdot(\mathbf{m}'_i \otimes \mathbf{m}'_j), \tag{10c}$$

where $\mathbf{D}'$ is the deviatoric part of $\mathbf{D}$, and the restriction of plastic incompressibility

$$\mathbf{D}_p\cdot\mathbf{I} = 0, \tag{11}$$

has been used.

The main constitutive problem is to develop physically meaningful constitutive equations for the rate of plastic deformation $\mathbf{D}_p$ and the plastic spin $\mathbf{W}_p$. To this end, it is noted that in view of eqns (5e) and (11) the restriction (7c) becomes

$$\mathbf{T}'\cdot\mathbf{D}_p \geqslant 0. \tag{12}$$

This means that the relaxation effects of plastic deformation are required to be dissipative. Motivated by the restrictions (11) and (12), a material that is plastically orthotropic is considered for which $\mathbf{D}_p$ takes the form

$$\mathbf{D}_p = \Gamma\bar{\mathbf{D}}_p, \tag{13a}$$

$$\begin{aligned}
\bar{\mathbf{D}}_p = {} & \frac{b_{11}}{2\mu}[J_m\mathbf{T}'\cdot(\mathbf{m}'_1 \otimes \mathbf{m}'_1)](\mathbf{m}'_1 \otimes \mathbf{m}'_1 - \tfrac{1}{3}m'_{11}\mathbf{I}) \\
& + \frac{b_{22}}{2\mu}[J_m\mathbf{T}'\cdot(\mathbf{m}'_2 \otimes \mathbf{m}'_2)](\mathbf{m}'_2 \otimes \mathbf{m}'_2 - \tfrac{1}{3}m'_{22}\mathbf{I}) \\
& + \frac{b_{33}}{2\mu}[J_m\mathbf{T}'\cdot(\mathbf{m}'_3 \otimes \mathbf{m}'_3)](\mathbf{m}'_3 \otimes \mathbf{m}'_3 - \tfrac{1}{3}m'_{33}\mathbf{I}) \\
& + \frac{b_{12}}{2\mu}[J_m\mathbf{T}'\cdot(\mathbf{m}'_1 \otimes \mathbf{m}'_2)](\mathbf{m}'_1 \otimes \mathbf{m}'_2 + \mathbf{m}'_2 \otimes \mathbf{m}'_1 - \tfrac{2}{3}m'_{12}\mathbf{I}) \\
& + \frac{b_{13}}{2\mu}[J_m\mathbf{T}'\cdot(\mathbf{m}'_1 \otimes \mathbf{m}'_3)](\mathbf{m}'_1 \otimes \mathbf{m}'_3 + \mathbf{m}'_3 \otimes \mathbf{m}'_1 - \tfrac{2}{3}m'_{13}\mathbf{I}) \\
& + \frac{b_{23}}{2\mu}[J_m\mathbf{T}'\cdot(\mathbf{m}'_2 \otimes \mathbf{m}'_3)](\mathbf{m}'_2 \otimes \mathbf{m}'_3 + \mathbf{m}'_3 \otimes \mathbf{m}'_2 - \tfrac{2}{3}m'_{23}\mathbf{I}),
\end{aligned} \tag{13b}$$

where Γ is a non-negative function that needs to be specified. The form (13b) was chosen to have the following properties. Firstly, the form (13b) causes the restriction (12) to be a quadratic function of the terms $[\mathbf{T}' \cdot (\mathbf{m}'_i \otimes \mathbf{m}'_j)]$ so that (12) will be satisfied if the coefficients b_{ij} are assumed to be non-negative constants

$$b_{ij} \geqslant 0. \tag{14}$$

Secondly, it is noted from (5d) that $J_m \mathbf{T}'$ is a function of elastic distortions only and in particular is independent of total volume change. In this same spirit the vectors $\mathbf{m}'_i$ have been used in (13b) instead of $\mathbf{m}_i$ because $\mathbf{m}'_i$ are also pure measures of elastic distortion. Thirdly, it is observed that even if b_{ij} are not equal, the rate of plastic deformation $\mathbf{D}_p$ and the stress $\mathbf{T}'$ will have the same principal directions when $\mathbf{m}'_i$ are orthogonal vectors that align with the principal directions of $\mathbf{T}'$. In this sense, the orthogonal triad $\mathbf{a}_i$ (Rubin, 1994) associated with the vectors $\mathbf{m}'_i$ can be related to the principal directions of plastic anisotropy discussed by Hill (1950, p. 318).

Here it is assumed that whenever the velocity gradient is a constant symmetric tensor (1) then the vectors $\mathbf{m}_i$ will asymptotically align themselves with the principal directions of $\mathbf{D}$. For this case the spin $\mathbf{W}$ vanishes and it follows from (1) and (9) that

$$\dot{\mathbf{m}}_i = (\mathbf{D} - \mathbf{D}_p)\mathbf{m}_i - \mathbf{W}_p \mathbf{m}_i. \tag{15}$$

Notice that the directions of $\mathbf{m}_i$ will remain constant if $\mathbf{W}_p$ vanishes and $\mathbf{m}_i$ are orthogonal vectors that align with the principal directions of $(\mathbf{D} - \mathbf{D}_p)$. Motivated by this observation the plastic spin $\mathbf{W}_p$ is specified in the form

$$\begin{aligned} \mathbf{W}_p = {} & \omega_{12}[\mathbf{D}_p \cdot (\mathbf{m}'_1 \otimes \mathbf{m}'_2)](\mathbf{m}'_1 \otimes \mathbf{m}'_2 - \mathbf{m}'_2 \otimes \mathbf{m}'_1) \\ & + \omega_{13}[\mathbf{D}_p \cdot (\mathbf{m}'_1 \otimes \mathbf{m}'_3)](\mathbf{m}'_1 \otimes \mathbf{m}'_3 - \mathbf{m}'_3 \otimes \mathbf{m}'_1) \\ & + \omega_{23}[\mathbf{D}_p \cdot (\mathbf{m}'_2 \otimes \mathbf{m}'_3)](\mathbf{m}'_2 \otimes \mathbf{m}'_3 - \mathbf{m}'_3 \otimes \mathbf{m}'_2), \end{aligned} \tag{16}$$

where ω_{12}, ω_{13}, ω_{23} are constants. This form for $\mathbf{W}_p$ will cause the triad $\mathbf{m}_i$ to tend to align itself with the principal directions of $\mathbf{D}_p$ (which causes $\mathbf{W}_p$ to vanish). Thus, whenever $\mathbf{W}$ vanishes, $\mathbf{D}$ is constant, and ω_{ij} are nonzero, plastic deformation will cause the triad $\mathbf{m}_i$ to rotate until $\mathbf{D}$ and $\mathbf{D}_p$ have the same principal directions. Obviously, the rate at which this transition takes place is controlled by the values of ω_{ij}. Also, notice that if $\mathbf{m}'_i$ in (16) are replaced by orthonormal vectors then (16) is similar to the form used by Raniecki and Mroz (1990) to model texture evolution in rigid–plastic materials.

It remains to specify a functional form for Γ and the evolution equations for the hardening variables κ and β_{ij}. Motivated by the work by Bodner (1987) and Rubin (1989) and neglecting thermal recovery of hardening, simple specific constitutive equations for elastic–viscoplastic materials are considered of the forms

$$\Gamma = \Gamma_0 \exp\left[-\frac{1}{2}\left(\frac{Z}{\sigma_e}\right)^{2n}\right], \quad \sigma_e^2 = \tfrac{3}{2}\mathbf{T}' \cdot \mathbf{T}', \tag{17a,b}$$

$$\dot{\kappa} = m_1 (J_m \mathbf{T}' \cdot \mathbf{D}_p)(Z_1 - \kappa), \tag{17c}$$

$$\dot{\beta}_{ij} = m_2 (J_m \mathbf{T}' \cdot \mathbf{D}_p)(Z_3 U_{ij} - \beta_{ij}), \tag{17d}$$

$$\mathbf{U} = \frac{\bar{\mathbf{D}}_p}{|\bar{\mathbf{D}}_p|}, \quad U_{ij} = \mathbf{U} \cdot (\mathbf{m}_i \otimes \mathbf{m}_j), \tag{17e,f}$$

$$Z = \kappa + \beta, \quad \beta = \boldsymbol{\beta} \cdot \mathbf{U} = \beta_{ij} m^{ir} U_{rs} m^{js}, \tag{17g,h}$$

where m^{ij} is the inverse of the metric m_{ij}. In (17): the function Γ causes yield-like behavior because it nearly vanishes for low values of the von Mises effective stress σ_e and it becomes significantly different from zero when σ_e attains values on the order of Z; Γ_0 is a positive constant; n is a positive constant which mainly controls strain-rate sensitivity; Z is a scalar measure of hardening; κ is a scalar measure of isotropic hardening; Z_1 is the saturated value of κ; β is a scalar measure of the effect of directional hardening $\boldsymbol{\beta}$; Z_3 is the saturated value of β; m_1 and m_2 are the constants controlling the rates of hardening. Also, it is recalled that the scalar measure β of directional hardening was introduced by Bodner (1985) to model the Bauschinger effect. In this sense, the present use of directional hardening can be considered an alternative to a more common kinematic hardening formulation.

Rate-insensitive elastic–plastic response can be obtained by (17) in the limit that the parameter n becomes large. Alternatively, it is possible to specify a yield function of the form

$$g = \left(\frac{\sigma_e}{Z}\right)^2 - 1, \tag{18}$$

and determine the scalar Γ by the loading conditions of the type discussed in Rubin (1994), but this will not be pursued further here.

3. UNIAXIAL STRESS

The influence of the orthotropic plastic response can be examined analytically by considering the simple case of uniaxial stress. To this end, it is assumed that the reference configuration is stress-free and at reference temperature θ_0. This means that in the reference configuration the metric m_{ij} is given by

$$m_{ij} = \delta_{ij}, \tag{19}$$

so that the vectors $\mathbf{m}_i$ form an orthonormal set. Specifically, consider uniaxial stress in the $\mathbf{e}_1$ direction with the fixed rectangular Cartesian base vectors $\mathbf{e}_i$ being parallel to $\mathbf{m}_i$. For this problem it can be shown that the velocity gradient may be specified in the form

$$\mathbf{L} = \mathbf{D} = \frac{\dot{a}}{a}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\dot{b}}{b}\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{\dot{c}}{c}\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{20}$$

with the initial conditions that

$$a(0) = b(0) = c(0) = 1. \tag{21}$$

Thus, a, b, c represent the stretches of material line elements in the coordinate directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively. Given a functional form for $a(t)$, the variables $b(t)$ and $c(t)$ are determined by the solution of the problem. In particular, with the help of the initial conditions it can be shown that for this problem m'_{ij} and $\mathbf{m}'_i$ may be represented in the forms

$$m'_{11} = a_m^2, \quad m'_{22} = m'_{33} = \frac{1}{a_m}, \quad \text{all other } m'_{ij} = 0, \tag{22a,b,c}$$

$$\mathbf{m}'_1 = a_m\mathbf{e}_1, \quad \mathbf{m}'_2 = \frac{1}{\sqrt{a_m}}\mathbf{e}_2, \quad \mathbf{m}'_3 = \frac{1}{\sqrt{a_m}}\mathbf{e}_3, \tag{22d,e,f}$$

where a_m is a function to be determined. Also, it follows from (10) and (20) that

$$\frac{\dot{a}}{a} = \frac{\dot{J}_m}{3J_m} + D'_{11}, \quad \frac{\dot{b}}{b} = \frac{\dot{J}_m}{3J_m} + D'_{22}, \quad \frac{\dot{c}}{c} = \frac{\dot{J}_m}{3J_m} + D'_{33}, \tag{23a,b,c}$$

and from (5d) and (22) that

$$T'_{11} = \frac{2\mu J_m^{-1}}{3}\left(\frac{a_m^3-1}{a_m}\right), \quad T'_{22} = T'_{33} = -\frac{\mu J_m^{-1}}{3}\left(\frac{a_m^3-1}{a_m}\right), \tag{24a,b}$$

$$\text{all other } T'_{ij} = 0, \tag{24c}$$

where D'_{ij} and T'_{ij} are the components of $\mathbf{D}'$ and $\mathbf{T}'$, respectively, relative to $\mathbf{e}_i$. Furthermore, it may be observed that for this case $\mathbf{W}_p$ vanishes and eqns (10c) yield three independent equations for D'_{ij} of the form

$$D'_{11} = \frac{\dot{a}_m}{a_m} + \frac{\Gamma}{18}\left(\frac{a_m^3-1}{a_m^3}\right)[4b_{11}a_m^6 + (b_{22}+b_{33})], \tag{25a}$$

$$D'_{22} = -\frac{1}{2}\frac{\dot{a}_m}{a_m} - \frac{\Gamma}{18}\left(\frac{a_m^3-1}{a_m^3}\right)[2b_{11}a_m^6 + (2b_{22}-b_{33})], \tag{25b}$$

$$D'_{33} = -\frac{1}{2}\frac{\dot{a}_m}{a_m} - \frac{\Gamma}{18}\left(\frac{a_m^3-1}{a_m^3}\right)[2b_{11}a_m^6 + (-b_{22}+2b_{33})]. \tag{25c}$$

Since the lateral boundary is stress-free it is required that

$$p = T'_{22}, \tag{26}$$

which with the help of (8) and (24b) yields an equation for J_m of the form

$$J_m = 1 + \frac{\mu}{3k}\left(\frac{a_m^3-1}{a_m}\right). \tag{27}$$

Now, by differentiating (27) with respect to time it can be deduced that

$$\frac{\dot{J}_m}{3J_m} = \left[\frac{\dfrac{\mu}{9k}\left(\dfrac{2a_m^3+1}{a_m}\right)}{1+\dfrac{\mu}{3k}\left(\dfrac{a_m^3-1}{a_m}\right)}\right]\frac{\dot{a}_m}{a_m}. \tag{28}$$

Thus, using (25a) and (28) the quantities J_m and D'_{11} can be eliminated from eqn (23a) to derive an equation for determining a_m of the form

$$\frac{\dot{a}_m}{a_m} = \left[\frac{1+\dfrac{\mu}{3k}\left(\dfrac{a_m^3-1}{a_m}\right)}{1+\dfrac{\mu}{9k}\left(\dfrac{5a_m^3-2}{a_m}\right)}\right]\left[\frac{\dot{a}}{a} - \frac{\Gamma}{18}\left(\frac{a_m^3-1}{a_m^3}\right)(4b_{11}a_m^6 + b_{22}+b_{33})\right], \tag{29}$$

with the initial condition $a_m(0) = 1$. Then, (25b,c), (28) and (29) can be substituted into (23b,c) to obtain equations for the stretches b and c.

It is of particular interest to note that when $\dot{a}/a$ is constant and the hardening variables κ and β have saturated, a steady-state solution exists for which a_m and Γ are independent of time. Specifically, it may be shown that for this steady-state solution

$$\frac{\dot{a}}{a} = \frac{\Gamma}{18}\left[\frac{a_m^3 - 1}{a_m^3}\right](4b_{11}a_m^6 + b_{22} + b_{33}), \tag{30a}$$

$$\frac{\dot{b}}{b} = -\frac{1}{2}\left[\frac{4b_{11}a_m^6 + 4b_{22} - 2b_{33}}{4b_{11}a_m^6 + b_{22} + b_{33}}\right]\frac{\dot{a}}{a}, \tag{30b}$$

$$\frac{\dot{c}}{c} = -\frac{1}{2}\left[\frac{4b_{11}a_m^6 - 2b_{22} + 4b_{33}}{4b_{11}a_m^6 + b_{22} + b_{33}}\right]\frac{\dot{a}}{a}, \tag{30c}$$

$$\dot{J}_m = 0. \tag{30d}$$

Thus, the stretching rates in the lateral direction will depend explicitly on the material constants b_{11}, b_{22}, b_{33}, which partially characterize the orthotropic plastic response.

4. SIMPLE SHEAR

In recent years it has been recognized that large deformation simple shear is a particularly helpful test of the physical applicability of elastic–plastic constitutive equations. Here, simple shear in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane is considered and the velocity gradient is specified by

$$\mathbf{L} = \dot{\gamma}\mathbf{e}_1 \otimes \mathbf{e}_2, \tag{31}$$

where $\gamma(t)$ is a measure of shear. It follows that the rate of deformation and spin tensors are given by

$$\mathbf{D} = \frac{\dot{\gamma}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad \mathbf{W} = \frac{\dot{\gamma}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{32a,b}$$

For simplicity, it is assumed that in the reference configuration the material is stress-free and at reference temperature θ_0 so that (19) holds and $\mathbf{m}_i$ are orthonormal vectors.

Before studying the elastic–viscoplastic response it is of interest to determine the purely elastic response to simple shear. To this end, it is noted that when $\mathbf{L}_p$ vanishes the evolution equation (9a) integrates to yield

$$\mathbf{m}_i = \mathbf{F}\mathbf{m}_i(0), \tag{33}$$

where $\mathbf{F}$ is the deformation gradient from the reference configuration to the present configuration. It follows that the elastic deformation tensor $\mathbf{B}_m$ becomes

$$\mathbf{B}_m = \mathbf{m}_i \otimes \mathbf{m}_i = \mathbf{F}[\mathbf{m}_i(0) \otimes \mathbf{m}_i(0)]\mathbf{F}^T = \mathbf{F}\mathbf{F}^T = \mathbf{B}. \tag{34}$$

Thus, for simple shear these expressions yield

$$\mathbf{F} = \mathbf{I} + \gamma\mathbf{e}_1 \otimes \mathbf{e}_2, \quad J_m = J = 1, \tag{35a,b}$$

$$\mathbf{B}_m = \mathbf{B} = \mathbf{I} + \gamma^2\mathbf{e}_1 \otimes \mathbf{e}_1 + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{35c}$$

$$\mathbf{T}' = \mu\left[\frac{2\gamma^2}{3}\mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{\gamma^2}{3}(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)\right]. \tag{35d}$$

In particular, notice that the normal stresses are of second order in γ and that T'_{22} is negative. Also, notice that the elastic deformation $\mathbf{B}_m$ and the deviatoric stress $\mathbf{T}'$ are independent of the initial orientation of $\mathbf{m}_i$ because the material is elastically isotropic. In contrast, the elastic–viscoplastic response that will be developed next exhibits a dependence on the initial orientation of $\mathbf{m}_i$ that is identified with texturing.

For the elastic response just discussed the values $\mathbf{m}_i(0)$ were allowed to have a general orientation relative to $\mathbf{e}_i$. However, for simplicity in interpreting the following elastic–viscoplastic response it is convenient to choose the shearing plane such that initially $\mathbf{m}_3 = \mathbf{e}_3$. It follows from the constitutive equations of Section 2 that $\mathbf{m}_1$ and $\mathbf{m}_2$ will remain in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane and that $\mathbf{m}_3$ will remain parallel to $\mathbf{e}_3$. Thus, the vectors $\mathbf{m}_i$ may be expressed in terms of their rectangular Cartesian components $F_{\mathrm{m}ij}$ relative to $\mathbf{e}_i$ such that

$$\mathbf{m}_i = F_{\mathrm{m}ij}\mathbf{e}_j, \quad F_{\mathrm{m}13} = F_{\mathrm{m}31} = F_{\mathrm{m}23} = F_{\mathrm{m}32} = 0. \tag{36a,b}$$

Now, letting $D_{\mathrm{p}ij}$ and $W_{\mathrm{p}ij}$ be the rectangular Cartesian components of $\mathbf{D}_\mathrm{p}$ and $\mathbf{W}_\mathrm{p}$ relative to $\mathbf{e}_i$, eqn (9a) may be rewritten in the form

$$\dot{F}_{\mathrm{m}11} = F_{\mathrm{m}11}(-D_{\mathrm{p}11}) + F_{\mathrm{m}12}(\dot{\gamma} - D_{\mathrm{p}12} - W_{\mathrm{p}12}), \tag{37a}$$

$$\dot{F}_{\mathrm{m}22} = F_{\mathrm{m}21}(-D_{\mathrm{p}12} + W_{\mathrm{p}12}) + F_{\mathrm{m}22}(-D_{\mathrm{p}22}), \tag{37b}$$

$$\dot{F}_{\mathrm{m}33} = F_{\mathrm{m}33}(-D_{\mathrm{p}33}), \tag{37c}$$

$$\dot{F}_{\mathrm{m}12} = F_{\mathrm{m}11}(-D_{\mathrm{p}12} + W_{\mathrm{p}12}) + F_{\mathrm{m}12}(-D_{\mathrm{p}22}), \tag{37d}$$

$$\dot{F}_{\mathrm{m}21} = F_{\mathrm{m}21}(-D_{\mathrm{p}11}) + F_{\mathrm{m}22}(\dot{\gamma} - D_{\mathrm{p}12} - W_{\mathrm{p}12}), \tag{37e}$$

where

$$D_{\mathrm{p}13} = D_{\mathrm{p}23} = 0, \quad W_{\mathrm{p}13} = W_{\mathrm{p}23} = 0, \tag{38a,b}$$

$$W_{\mathrm{p}12} = \omega_{12}[\mathbf{D}_\mathrm{p} \cdot (\mathbf{m}_1' \otimes \mathbf{m}_2')](F_{\mathrm{m}11}F_{\mathrm{m}22} - F_{\mathrm{m}21}F_{\mathrm{m}12}). \tag{38c}$$

Detailed expressions for the rectangular Cartesian components T'_{ij} of $\mathbf{T}'$ as well as for $D_{\mathrm{p}ij}$ can be derived in a straight-forward manner and are not presented here. Notice that since J_m is unity the pressure vanishes and the Cauchy stress $\mathbf{T}$ is deviatoric so that $T_{ij} = T'_{ij}$.

In general, the rotation of the triad $\mathbf{m}_i$ is influenced by the value of ω_{12}. For large values of ω_{12} the triad will try to align itself close to the principal directions of $\mathbf{D}_\mathrm{p}$ but since $\mathbf{m}_i$ are not orthogonal vectors and $\mathbf{W}$ is nonzero it will not align itself exactly, even asymptotically. In order to study the texturing effect of the rotation of $\mathbf{m}_i$ different values of ω_{12} are considered which control the rate of rotation of $\mathbf{m}_i$, and different initial orientations of $\mathbf{m}_i$ relative to $\mathbf{e}_i$ are considered which control the orientation of the shearing direction relative to the directions of anisotropy. To this end, it is assumed that initially $\mathbf{m}_i$ are given by the formulas

$$\mathbf{m}_1 = \cos\phi_0\mathbf{e}_1 + \sin\phi_0\mathbf{e}_2, \quad \mathbf{m}_2 = -\sin\phi_0\mathbf{e}_1 + \cos\phi_0\mathbf{e}_2, \quad \mathbf{m}_3 = \mathbf{e}_3, \tag{39a,b,c}$$

where ϕ_0 is the initial angle that $\mathbf{m}_1$ makes with the $\mathbf{e}_1$ direction, measured positive in the counterclockwise direction. Next, to track the rotation of $\mathbf{m}_i$ as the deformation progresses the angle ϕ associated with the orientation of $\mathbf{m}_1$ relative to $\mathbf{e}_1$ is defined by the formulas

$$\cos\phi = \frac{F_{\mathrm{m}11}}{|\mathbf{m}_1|}, \quad \sin\phi = \frac{F_{\mathrm{m}12}}{|\mathbf{m}_1|}, \quad -\pi < \phi \leqslant \pi. \tag{40a,b,c}$$

For the simulations presented here $\dot{\gamma}$ is taken to be constant during finite time intervals so that

$$\dot{\gamma} = \pm 1.0 \times 10^{-4}\ \mathrm{s}^{-1}, \tag{41}$$

and eqns (17c,d), (37) and (41) are integrated using the initial conditions

$$\gamma = 0, \qquad F_{m11} = \cos\phi_0, \quad F_{m12} = \sin\phi_0, \tag{42a,b,c}$$

$$F_{m21} = -\sin\phi_0, \quad F_{m22} = \cos\phi_0, \quad F_{m33} = 1, \tag{42d,e,f}$$

$$\kappa = \kappa_0, \qquad \beta_{ij} = 0. \tag{42g,h}$$

Furthermore, the material constants are specified by the same values as those used in the simulations of a typical (but not specific) material discussed in Rubin (1987b) so that

$$\mu = 44.0\ \text{GPa}, \qquad \Gamma_0 = 10^8\ \text{s}^{-1}, \qquad n = 1.0, \tag{43a,b,c}$$

$$\kappa_0 = 1.7\ \text{GPa}, \qquad Z_1 = 2.0\ \text{GPa}, \qquad Z_3 = 1.0\ \text{GPa}, \tag{43d,e,f}$$

$$m_1 = 100.0\ \text{GPa}^{-1}, \quad m_2 = 4000.0\ \text{GPa}^{-1}. \tag{43g,h}$$

For most of the simulations plastic relaxation is taken to be isotropic in the sense that the values of b_{ij} are equal

$$b_{ij} = 1 \quad \text{for isotropic plastic relaxation.} \tag{44}$$

However, a simple case where b_{ij} are specified by

$$b_{11} = 0.5, \quad b_{22} = 0.75, \quad \text{all other } b_{ij} = 1, \quad \text{for anisotropic plastic relaxation} \tag{45}$$

will be considered to examine the influence of anisotropic plastic relaxation. In this regard, it is observed from (13) that even when b_{ij} are equal, plastic relaxation does not remain isotropic because directional effects develop as $\mathbf{m}'_i$ distort and as the directional hardening parameter β_{ij} becomes nonzero.

At this point it should be mentioned that the functional form for Γ causes the differential equations to be stiff. Special methods of the type developed by Rubin (1989) for another class of equations of this type have not yet been developed for the equations presented here. Therefore, here the equations were integrated using small time steps and a fourth-order Runge–Kutta numerical integration procedure. To examine the accuracy of the integration the numerical values of $J_m - 1$ and $\mathbf{T}' \cdot \mathbf{I}$ were checked and found to remain negligibly small (note that simple shear is isochoric). Also, the time step was halved for a single simulation to verify that acceptable convergence had been obtained.

Figure 1 shows the effects of changing the value of ω_{12} while keeping the initial orientation of $\mathbf{m}_i$ the same with $\phi_0 = 0$. Notice from Fig. 1(b) that for $\omega_{12} = -5$ the triad rotates clockwise until ϕ saturates at a value close to a value of $-\pi/4$, whereas for $\omega_{12} = 5$ the triad rotates counterclockwise until ϕ saturates at a value close to a value of $\pi/4$. Thus, the triad tries to align itself with the principal directions of $\mathbf{D}$ which correspond to $\phi = \pm\pi/4$ but it cannot align itself exactly with these principal directions. For $\omega_{12} = -1$ the triad rotates more slowly and ϕ does not have a chance to saturate. For $\omega_{12} = 0$ the triad will continue to rotate clockwise indefinitely due to the nonzero value of $\mathbf{W}$. Also, for $\omega_{12} = 1$ the tendency for the triad to rotate clockwise due to $\mathbf{W}$ is nearly cancelled by the tendency to rotate counterclockwise due to plastic relaxation effects. As far as the stress response is concerned note that the texture effect of the triad rotation has negligible influence on the shear stress T_{12} and the normal stress T_{33}, but it significantly influences the other normal stresses. In this sense, the texturing effect of triad rotation mainly influences second-order effects in simple shear. In particular, it is observed that by changing the sign of ω_{12} it is possible to change the sign of the normal stresses T_{11} and T_{22} as they transition to their steady-state values. For ω_{12} nonzero the normal stresses asymptotically approach the same steady-state values, independent of the magnitude of ω_{12} (the response for $\omega_{12} = -1$ has not yet reached steady state). However, the steady-state values of the normal stresses for ω_{12} equal to zero are different from those for ω_{12} nonzero. Further, in this regard, it is

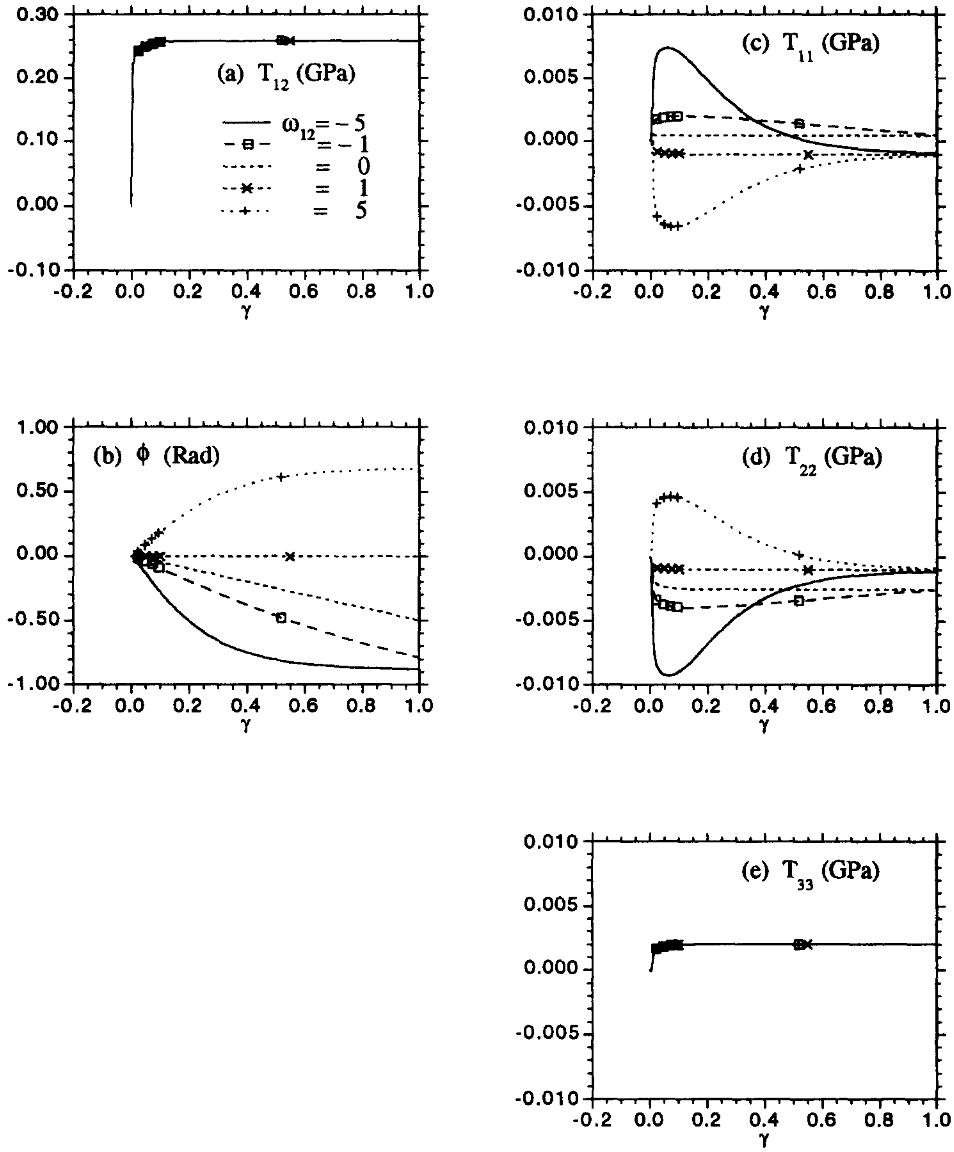

Fig. 1. Simple shear: the effect of changing ω_{12}. For all cases $\phi_0 = 0$.

recalled from Khen and Rubin (1992) that the steady-state values of the normal stresses can be influenced by second-order elastic effects that can be modeled by including both pure measures of elastic distortion α_1 and α_2 in the Helmholtz free energy.

The normal stress T_{22} in simple shear is usually observed to be compressive. For this reason the response for the value $\omega_{12} = -5$ will be used as a reference for the analysis of the present model even though the transition to steady state associated with this value may yield slightly exaggerated values of normal stress. It is also worth mentioning that the oscillatory response shown in Fig. 1(d) is similar to that observed by Montheillet *et al.* (1984) for simple shear at high temperature. This suggests that the oscillatory behavior of the normal stress may be due to texturing that is enhanced at high temperature. A similar result was indicated by the analysis of Dafalias and Rashid (1989, p. 235) using a different set of equations.

Figure 2 shows the effect of shearing in different material directions. This is accomplished by changing the initial orientation of $\mathbf{m}_i$ while keeping $\omega_{12} = -5$. Notice from Fig. 2(b) that for $\phi_0 \in [-\pi/2, \pi/4]$ the triad $\mathbf{m}_i$ rotates towards the same saturated value of ϕ near $-\pi/4$, whereas for $\phi_0 = \pi/2$ the triad rotates counterclockwise towards a

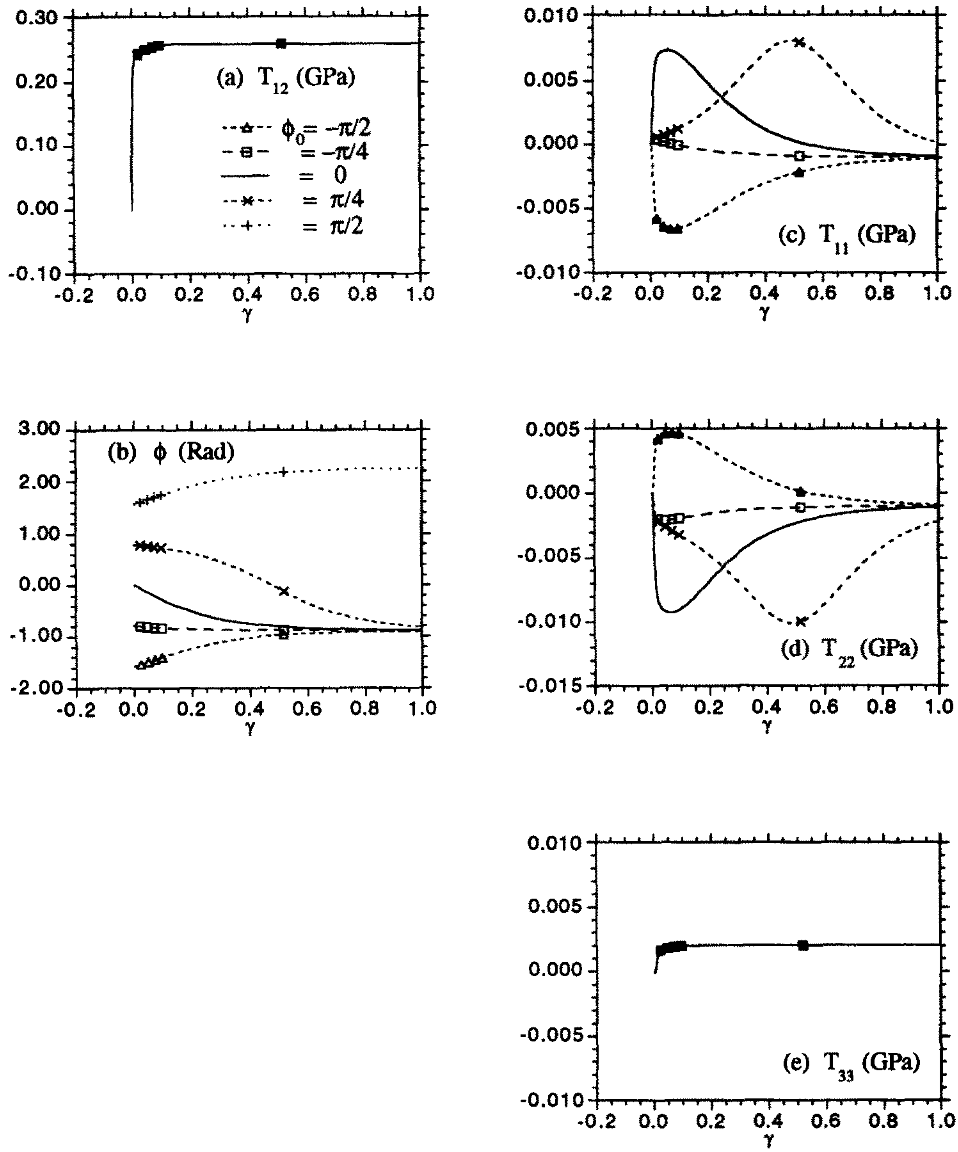

Fig. 2. Simple shear: the effect of changing ϕ_0. For all cases $\omega_{12} = -5$.

saturated value of ϕ near $3\pi/4$. Again, the shear stress T_{12} and the normal stress T_{33} are relatively uninfluenced by the changes in ϕ_0. However, the normal stresses T_{11} and T_{22} are significantly influenced by the change in material orientation, except for the fact that the response to $\phi_0 = \pm\pi/2$ is the same. In particular, the normal stresses T_{11} and T_{22} change sign when ϕ_0 is changed from 0 to $\pi/2$ in a similar manner to the change in sign observed in Figs 1(c) and (d) when ω_{12} changes sign.

Figure 3 shows the response to a large deformation cycle with $\phi_0 = 0$ and $\omega_{12} = -5$. Note the hysteretic nature of the orientation of the triad and the normal components of stress. Figure 4 shows the influence of anisotropic plastic relaxation by comparing the responses predicted by the isotropic case associated with the specification (44) and the anisotropic case associated with the specification (45). It is observed that anisotropic plastic relaxation significantly influences the normal stresses but not the shear stress T_{12} or the orientation of the triad. The influence of anisotropic plastic relaxation on a small deformation cycle is shown in Fig. 5, with the expanded views in Figs 5(b) and (d). In particular from Figs 5(a) and (c) it is observed that the shear stress is nearly the same for both the isotropic and the anisotropic materials. Also, note the characteristic sharp elastic–plastic

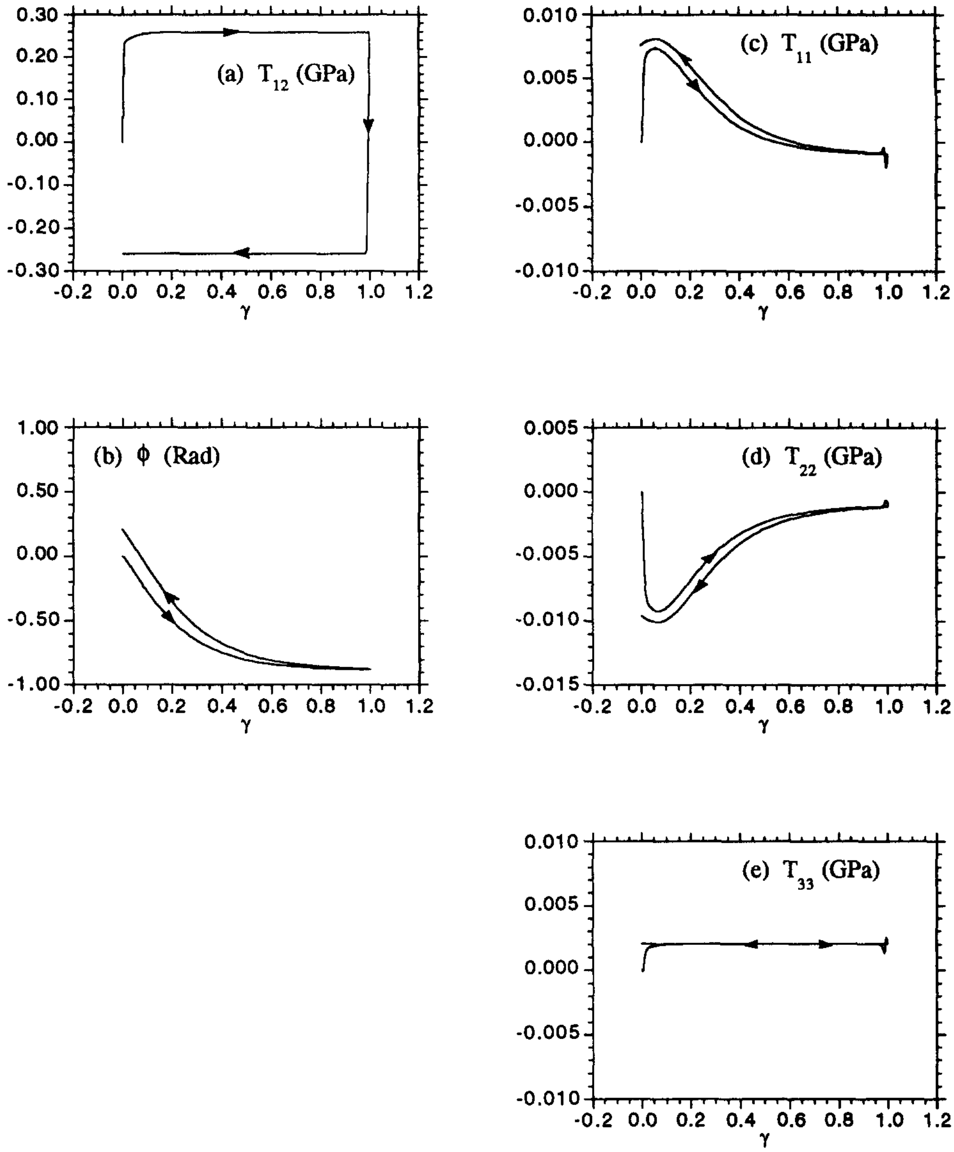

Fig. 3. Simple shear: a large deformation cycle with $\phi_0 = 0$ and $\omega_{12} = -5$.

transition for unloading and reloading in the same direction near $\gamma = 0.08$, and the Bauschinger effect in the cycle which starts unloading from $\gamma = 0.10$.

Recall that the expression (13b) for the plastic deformation rate was chosen so that the plastic relaxation would remain dissipative (12) even when plastic relaxation was orthotropic. However, when b_{ij} are specified by the isotropic form (44) the expression (13b) retains dependence on $\mathbf{m}'_i$ which is not usually present in isotropic constitutive equations for large deformation plasticity. In order to examine the influence of this dependence on $\mathbf{m}'_i$ it is possible to consider a modified form for $\bar{\mathbf{D}}_p$ which is more common

$$\bar{\mathbf{D}}_p = \frac{J_m \mathbf{T}'}{2\mu}, \quad \mathbf{T}' \cdot \mathbf{D}_p = \Gamma \frac{J_m \sigma_e^2}{3\mu} \geqslant 0. \tag{46a,b}$$

For this modified constitutive equation the expression (16) for the plastic spin $\mathbf{W}_p$ is retained, which allows $\mathbf{W}_p$ to vanish when ω_{ij} vanish.

Figure 6 compares the response of the model with $\bar{\mathbf{D}}_p$ specified by (13b) and b_{ij} specified by (44), with the modified model specified by (46a). For each case ϕ_0 vanishes and ω_{12} is

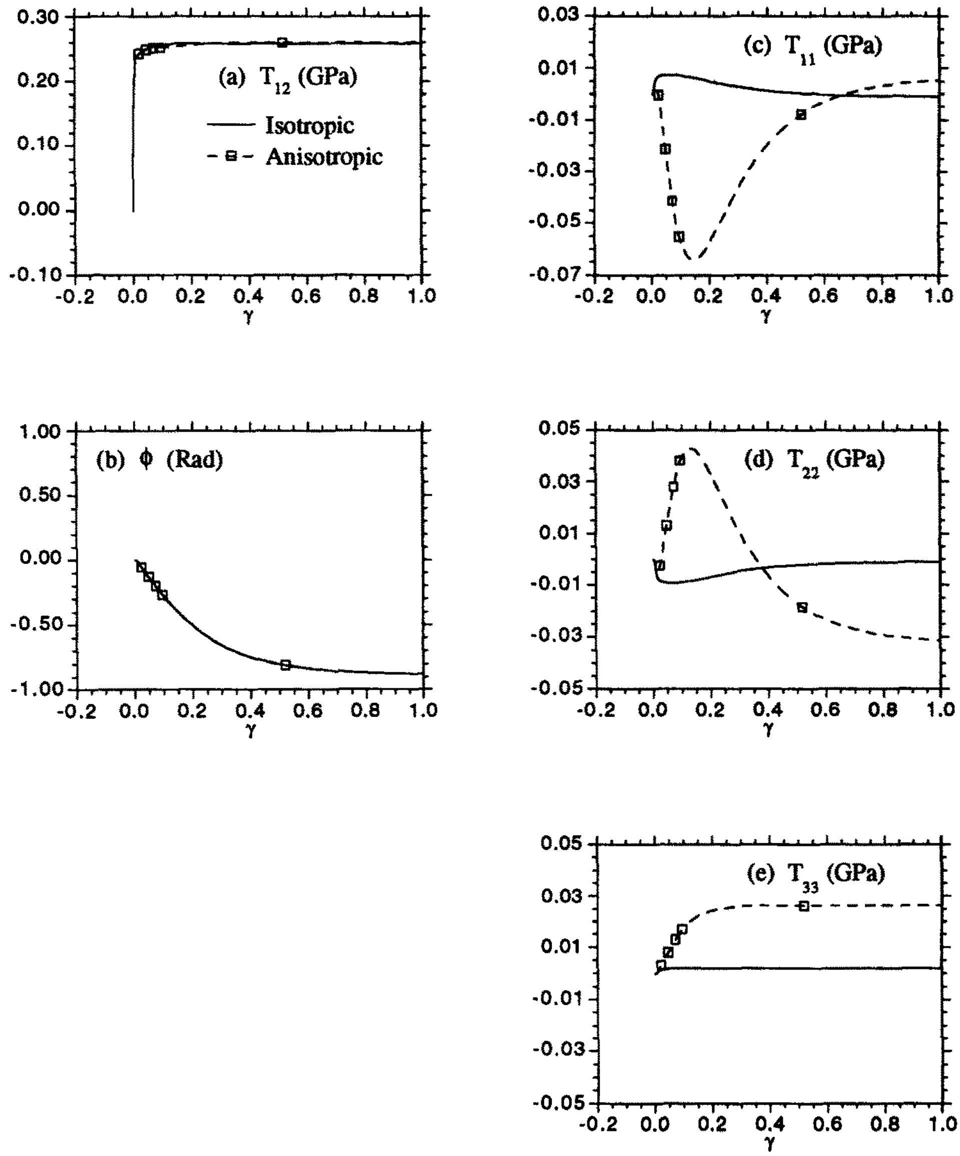

Fig. 4. Simple shear: the influence of anisotropic plastic relaxation with $\phi_0 = 0$ and $\omega_{12} = -5$.

either 0 or -5. Notice that the shear stress T_{12} and the triad rotation are nearly uninfluenced by the modified form (46a) but that the normal stresses are significantly influenced by this form. Most importantly, is the fact that the present model for elastic–viscoplastic response predicts physically reasonable response in the presence of directional hardening, even when plastic relaxation is specified in the simple form (46a) with vanishing plastic spin.

5. ISOCHORIC EXTENSION

The constitutive equation (13) for the rate of plastic deformation $\mathbf{D}_p$ and eqn (16) for the plastic spin $\mathbf{W}_p$ were chosen so that whenever the velocity gradient $\mathbf{L}$ is symmetric and constant (1) the triad $\mathbf{m}_i$ will asymptotically align itself with the principal directions of $\mathbf{D}$. In order to examine a simple example of this response it is possible to consider isochoric extension which is specified by

$$\mathbf{L} = \frac{\dot{a}}{a}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \tfrac{1}{2}\mathbf{e}_2 \otimes \mathbf{e}_2 - \tfrac{1}{2}\mathbf{e}_3 \otimes \mathbf{e}_3), \tag{47}$$

where the stretch a and the extension E in the $\mathbf{e}_1$ direction are determined by the equations

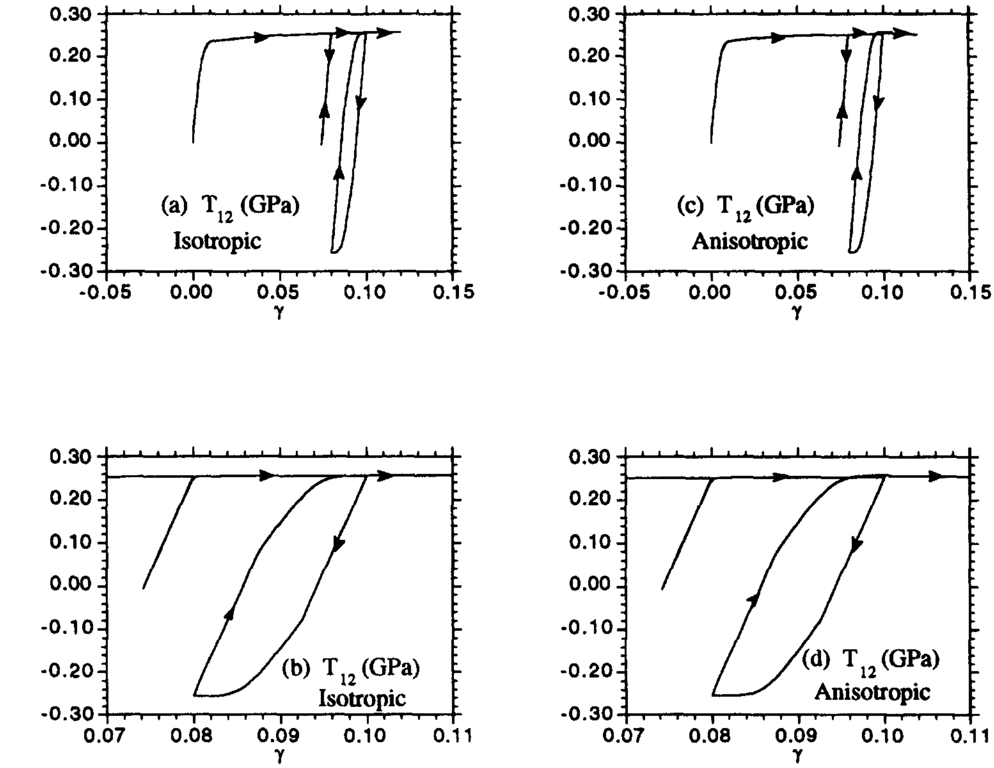

Fig. 5. Simple shear: a small deformation cycle showing the influence of anisotropic plastic relaxation with $\phi_0 = 0$ and $\omega_{12} = -5$.

$$\frac{\dot{a}}{a} = 1.0 \times 10^{-4}\ \mathrm{s}^{-1}, \quad E = a - 1. \tag{48a,b}$$

The fixed directions $\mathbf{e}_i$ are chosen so that the initial values of $\mathbf{m}_i$ are given by (39) and the angle ϕ is defined by (40). It then follows that $\mathbf{m}_i$ are determined by (36) and the evolution equations

$$\dot{F}_{\mathrm{m}11} = F_{\mathrm{m}11}\left(\frac{\dot{a}}{a} - D_{\mathrm{p}11}\right) - F_{\mathrm{m}12}(D_{\mathrm{p}12} + W_{\mathrm{p}12}), \tag{49a}$$

$$\dot{F}_{\mathrm{m}22} = F_{\mathrm{m}21}(-D_{\mathrm{p}12} + W_{\mathrm{p}12}) - F_{\mathrm{m}22}\left(\frac{1}{2}\frac{\dot{a}}{a} + D_{\mathrm{p}22}\right), \tag{49b}$$

$$\dot{F}_{\mathrm{m}33} = -F_{\mathrm{m}33}\left(\frac{1}{2}\frac{\dot{a}}{a} + D_{\mathrm{p}33}\right), \tag{49c}$$

$$\dot{F}_{\mathrm{m}12} = F_{\mathrm{m}11}(-D_{\mathrm{p}12} + W_{\mathrm{p}12}) - F_{\mathrm{m}12}\left(\frac{1}{2}\frac{\dot{a}}{a} + D_{\mathrm{p}22}\right), \tag{49d}$$

$$\dot{F}_{\mathrm{m}21} = F_{\mathrm{m}21}\left(\frac{\dot{a}}{a} - D_{\mathrm{p}11}\right) - F_{\mathrm{m}22}(D_{\mathrm{p}12} + W_{\mathrm{p}12}), \tag{49e}$$

where (38) holds. Equations (17c,d), (48) and (49) are integrated subject to the initial conditions (42b–h) and

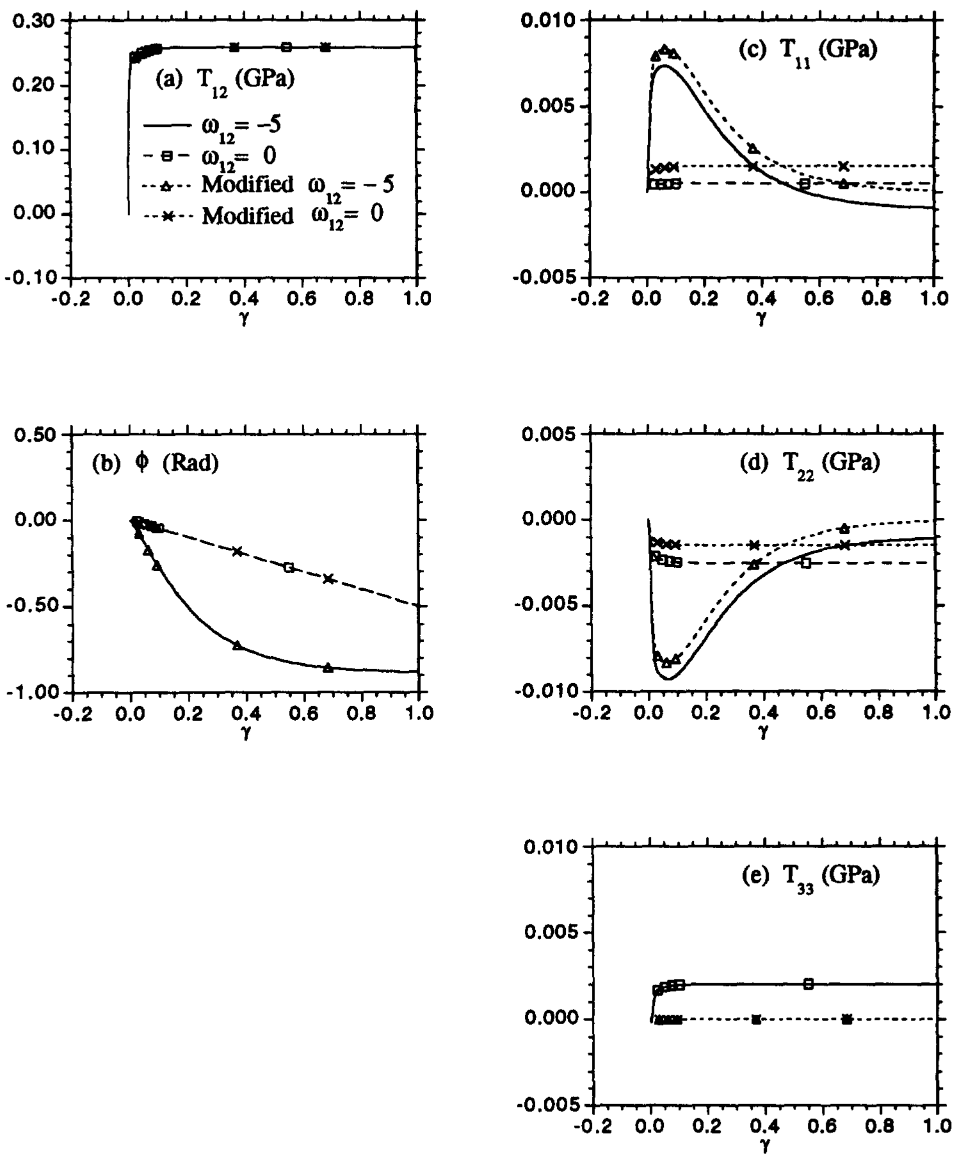

Fig. 6. Simple shear: the influence of the modified constitutive equation for $\mathbf{D}_p$.

$$a(0) = 1, \tag{50}$$

and the material constants are specified by (43). Also, b_{ij} is specified by (44) for the isotropic case and (45) for the anisotropic case. In all cases $\mathbf{D}_p$ is specified by (13) and $\mathbf{W}_p$ is specified by (16) with $\omega_{12} = -5$.

Figures 7 and 8 show the response to isochoric extension for isotropic and anisotropic plastic relaxation, respectively. For the isotropic case the stresses T_{22} and T_{33} are equal and the stresses T_{11}, T_{22}, T_{33} are relatively uninfluenced by the initial orientation of the triad. However, the shear stress T_{12} is nonzero when the triad is not oriented parallel to the principal directions of $\mathbf{D}$. Notice also, from Figs 7(c), (d) and (e) that the response for ϕ_0 equal to 0 or $\pi/2$ is the same. Figure 8 shows that for anisotropic plastic relaxation the stresses T_{22} and T_{33} are no longer the same. Also, the stresses T_{11}, T_{22} and T_{33} are now influenced by the initial orientation of the triad. Notice from Figs 8(c), (d) and (e) that the responses for ϕ_0 equal to 0 and $\pi/2$ are no longer equal.

From Figs 7(b) and 8(b) it is observed that when the triad is not initially oriented parallel to the principal directions of $\mathbf{D}$ and ϕ_0 is not equal to zero, the triad tends to rotate

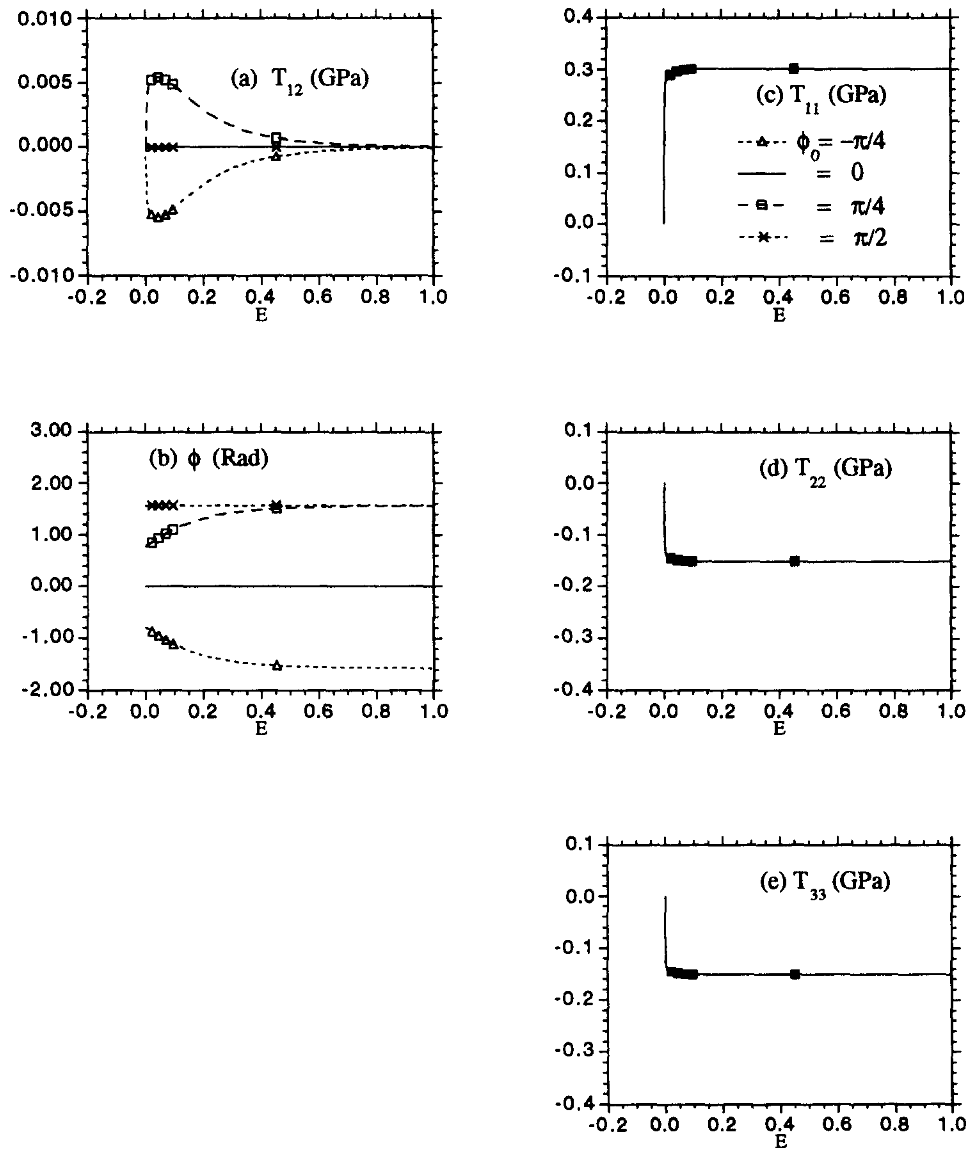

Fig. 7. Isochoric extension: the response for isotropic plastic relaxation.

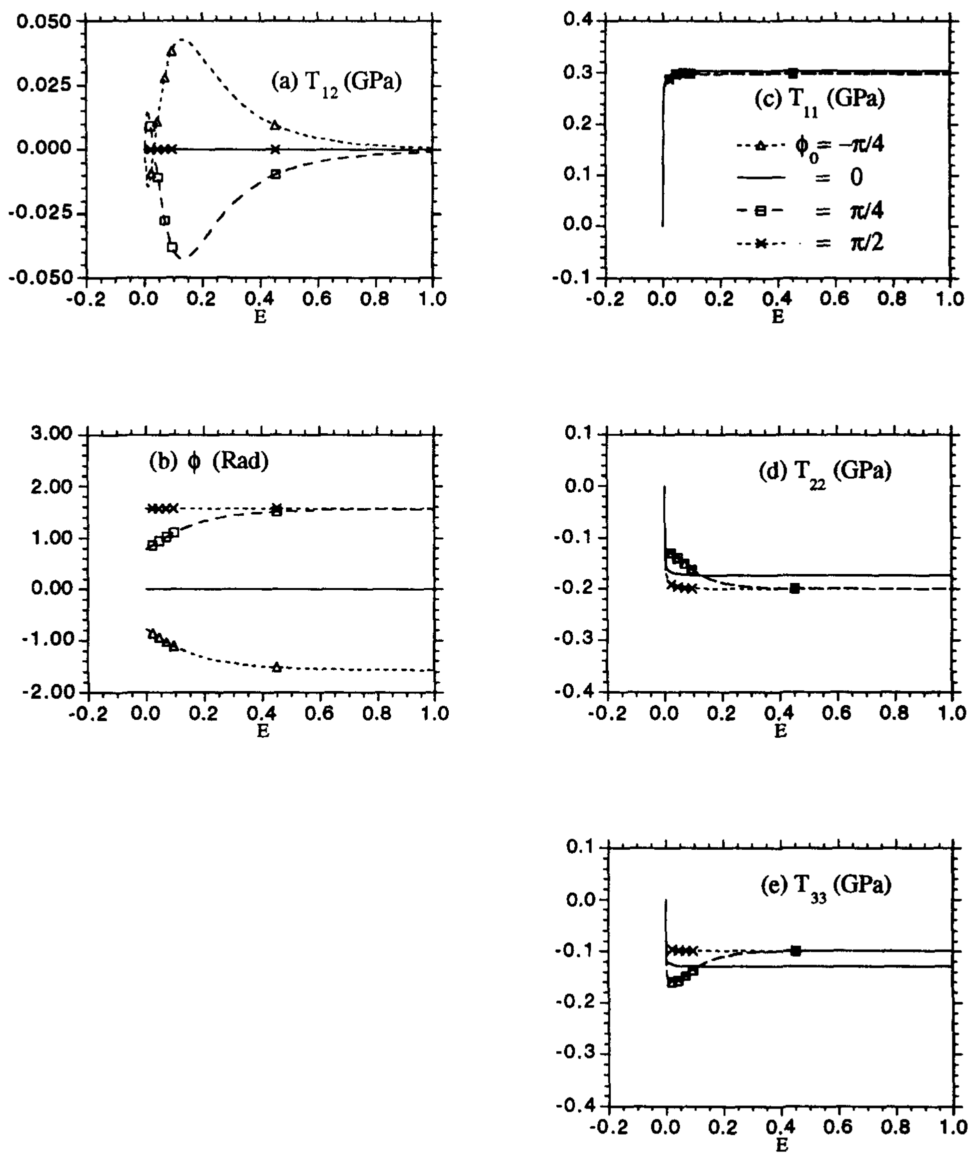

Fig. 8. Isochoric extension: the response for anisotropic plastic relaxation.

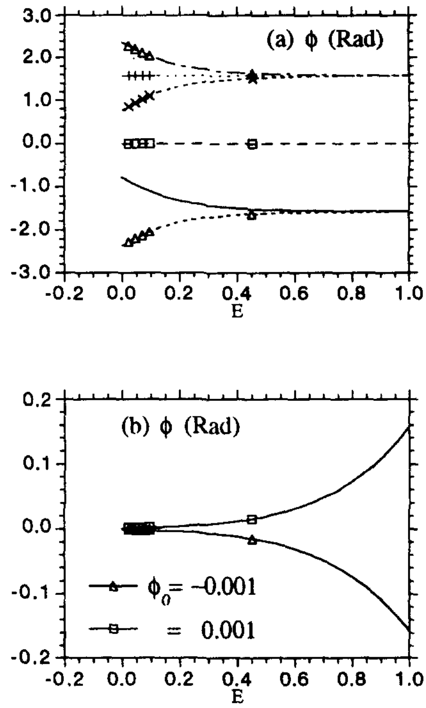

Fig. 9. Isochoric extension: the rotation of the triad for isotropic plastic relaxation.

so that ϕ equals either $\pi/2$ or $-\pi/2$. This causes $\mathbf{m}_1$ to align itself with the direction associated with contraction of a line element. Figure 9 shows this response more clearly for isotropic plastic relaxation. In particular, notice from Fig. 9(a) that when ϕ_0 is positive and in the range $(0, \pi)$ then ϕ tends to the value $\pi/2$, whereas when ϕ_0 is negative and in the range $(0, -\pi)$ then ϕ tends to the value $-\pi/2$. Figure 9(b) emphasizes the special nature of the initial orientation associated with $\phi_0 = 0$ (and similarly $\phi_0 = \pi$) by showing the response for $\phi_0 = \pm 0.001$ rad. Additional simulations of isochoric contraction ($\dot{a}/a < 0$) in the $\mathbf{e}_1$ direction indicated that $\mathbf{m}_1$ again tended to align itself with the direction of contraction of a line element (which for this case was the $\mathbf{e}_1$ direction with $\phi = 0$).

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